Homological Algebra 4.1 & 4.2

Zero

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Outline

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2. Semisimple Rings
3. von Neumann Regular Rings
4. Summary

Main theorems

Theorem

Ring R is semisimple
$\iff$ Every left / right R-module is projective
$\iff$ Every left / right R module is injective

Theorem

Ring R is von Neumann regular
$\iff$ Every right R-module is flat

Definition of semisimple module

Definition

R: a ring
$M \in {}_R\mathrm{Mod}$
M is simple (irreducible), if: $M \neq \{0\}$ has no non-trivial submodule
M is semisimple (completely reducible), if: it is a direct sum of simple modules.

$\{0\} = \oplus_{i \in \emptyset} S_i$ is semisimple, but not simple.

Semisimple module iff submodule direct summand

Proposition

R: a ring

$M \in {}_R\mathrm{Mod}$

M is semisimple $\iff$ every submodule is a direct summand.

Prop: $M \in {}_R\mathrm{Mod}$ is semisimple $\Leftrightarrow$ every submodule is a direct summand.

Proof: "$\Rightarrow$": M semisimple, then $M = \bigoplus_{j \in J} S_j$, S_j simple.

Take $N \subseteq M$ is any submodule. If $N = M$, done.

If $N \subsetneq M$: we denote $S_I = \bigoplus_{j \in I} S_j$ for $I \subseteq J$.

Let $\Phi = \{ I \mid S_I \cap N = \{0\} \} \neq \phi$, by Zorn's,

there is a maximal $I \in \Phi$, denote as I_{max}.

- We claim: $M = N \oplus S_{I_{max}}$. Since $N \cap S_{I_{max}} = \{0\}$, it is enough to show $M = N + S_{I_{max}}$, it is enough to show $S_j \subseteq N + S_{I_{max}}$ for $\forall j \in J$. ($M = \bigoplus_{j \in J} S_j$)

- If $j \in I_{max}$, $S_j \subseteq N + S_{I_{max}}$ ✓.
- If $j \notin I_{max}$, set $I' = I_{max} \cup \{j\}$, bigger than I_{max}, so $S_{I'} \cap N \neq \{0\}$.

  Take $0 \neq n \in S_{I'} \cap N$, $n \in S_{I'} = \left(\bigoplus_{i \in I_{max}} S_i \right) \oplus S_j = S_{I_{max}} \oplus S_j$

  so there is $s_I \in S_{I_{max}}$, $s_j \in S_j$, s.t. $n = s_I + s_j$.

  $0 \neq s_j = n - s_I \in (N + S_{I_{max}}) \cap S_j$, ($s_j \neq 0$, otherwise $n = s_I \in S_{I_{max}} \cap N = \{0\}$)

  , S_j is simple, has no non-trivial submodule,

  while $(N + S_{I_{max}}) \cap S_j \subseteq S_j$ ($s_j \in$), $(N + S_{I_{max}}) \cap S_j = S_j$,

  $\Rightarrow S_j \subseteq N + S_{I_{max}}$.

"$\Leftarrow$": For $\forall N \subseteq M$ a submodule, take any $0 \neq x \in N$,

construct $\Phi = \{ Z \subseteq N \text{ submodule} \mid x \notin Z \} \neq \phi$ $[\{0\} \in \Phi]$,

by Zorn's, there is a maximal, denote as Z_{max}.

- Z_{max} is a submodule of M, by hypothesis, $M = Z_{max} \oplus \overline{Z_{max}}$ for some submodule $\overline{Z_{max}}$. $Z_{max} \subseteq N \subseteq M$, then

  $N = Z_{max} \oplus (N \cap \overline{Z_{max}}) =: Z_{max} \oplus Y$. (P51, coro 2.24)

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- We claim: Y is simple. Otherwise, $\{0\} \neq Y' \subsetneq Y$ non-trivial submodule,

  Y' is a direct summand of Y, say, $Y = Y' \oplus Y''$,

  then $N = Z_{max} \oplus Y' \oplus Y''$. $x \notin Z_{max} \oplus Y'$ or $x \notin Z_{max} \oplus Y''$

  hence, $Z_{max} \oplus Y' \in \Phi$ or $Z_{max} \oplus Y'' \in \Phi$. contradiction! (otherwise, $x \in (Z_{max} \oplus Y') \cap (Z_{max} \oplus Y'') = Z_{max}$)

- To show M is semisimple. Let $S = \{ S_k \mid S_k \subseteq M \text{ a simple submodule} \} \neq \phi$

  Construct $\Delta = \{ (S_k)_{k \in K} \subseteq S \mid \bigoplus_{k \in K} S_k \text{ generated by } (S_k)_{k \in K} \} \neq \phi$,

  by Zorn's, there is a maximal, denote as $(S_k)_{k \in K}$.

  Then $D := \bigoplus_{k \in K} S_k \subseteq M$ a submodule, by hypothesis, $M = D \oplus E$.

  - If $E = \{0\}$, done. ✓
  - If $E \neq \{0\}$, $S \subseteq E$ is a simple submodle, $(S_k)_{k \in K} \cup S \in \Delta$, is bigger, constraction!

Submodule and quotient module

Corollary

Every submodule and every quotient module of a semisimple module M is semisimple.

Corollary: Every submodule and every quotient module of a semisimple module is semisimple.

Proof: M semisimple, $N \subseteq M$ a submodule,

~~$\Rightarrow M = N \oplus N$~~. $\Rightarrow M = N \oplus Q$ for some Q.

- For any $S \subseteq N$, S is a submodule of M,

  $\Rightarrow M = S \oplus \bar{S}$, $S \subseteq N \subseteq M$,

  $\Rightarrow N = S \oplus (N \cap \bar{S}) \Rightarrow S$ is a direct summand, S is arbitrary,

  $\Rightarrow N$ is semisimple.

$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$, $M/N \cong Q$,

Q is semisimple $\Rightarrow M/N$ is semisimple.

$\square$.

Direct sum of left ideals

Lemma

If a ring R is a direct sum of left ideals, say, $R = \oplus_{i \in I} L_i$, then only finitely many L_i are non-zero.

Lemma: If a ring R is a direct sum of left ideals, say, $R = \bigoplus_{i \in I} L_i$, then only finitely many L_i are non-zero.

Proof: Express $1 \in R$ as:

$$1 = e_1 + \cdots + e_n, \text{ only finite sum,}$$

$$e_1 \in L_1, \ \cdots, \ e_n \in L_n, \ 1, \cdots, n \in I$$

We claim only $L_1, \cdots, L_n$ are non-zero.

In fact, if $a \in L_{n+1}$,

$$a = a \cdot 1 = ae_1 + \cdots + ae_n \in L_{n+1} \cap (L_1 \oplus \cdots \oplus L_n) = \{0\},$$

$$\Rightarrow L_{n+1} = \{0\}.$$

□.

Definition of semisimple ring

Definition

A ring R is semisimple, if: it is semisimple as a left R-module; if: it is a direct sum of minimal left ideals.

Definition

R: a ring
$L \subseteq R$ is the minimal ideal, if: $L \neq \{0\}$, and there is no left ideal J, s.t., $\{0\} \subsetneqq J \subsetneqq L$.

Examples

- (ii) A ring is left semisimple if and only if it is right semisimple. See, Advanced Modern Algebra, P563, Corollary 8.57. Proved by (i).
- (v) A finite direct product of fields is semisimple.

The main theorem

Theorem

TFAE:

(i) R is semisimple

(ii) Every left / right R-module M is a semisimple module

(iii) Every left / right R-module M is injective

(iv) Every short exact sequence of left / right R-module splits

(v) Every left / right R-module M is projective

TFAE: (i) R is semisimple.

(ii) Every $M \in {}_R\mathrm{Mod}$ is a semisimple module.

(iii) Every $M \in {}_R\mathrm{Mod}$ is injective.

(iv) Every short exact sequence in ${}_R\mathrm{Mod}$ splits.

(v) Every $M \in {}_R\mathrm{Mod}$ is projective.

Proof: (i) $\Rightarrow$ (ii): R semisimple $\Rightarrow R = \bigoplus_{i \in I} L_i$, L_i minimal ideal,

$\Rightarrow \forall$ F free module, $F = R^n = R \oplus \cdots \oplus R$
$= (\oplus L_i) \oplus \cdots \oplus (\oplus L_i)$
is semisimple

$\Rightarrow \forall M \in {}_R\mathrm{Mod}$, $M = F/Q \Rightarrow M$ semisimple.

(ii) $\Rightarrow$ (iii): For any $0 \to E \to B \to C \to 0$ exact,

$E, B, C \in {}_R\mathrm{Mod}$, $E, \underline{B}, C$ semisimple,

$E \subseteq B$ a submodule, $\Rightarrow B = E \oplus \bar{E}$ for some $\bar{E}$, $\Rightarrow$ that sequence splits

$\Rightarrow E$ is injective.

(iii) $\Rightarrow$ (iv): For any $0 \to A \to B \to C \to 0$,

$A, B, C \in {}_R\mathrm{Mod}$, $\underline{A}, B, C$ injective,

$A \subseteq B$ submodule $\Rightarrow A$ is a direct summad of B

(P116: Coro 3.27)

$\Rightarrow$ that sequence splits.

(iv) $\Rightarrow$ (v): For any $M \in {}_R\mathrm{Mod}$, any sequence

$0 \to A \to B \to M \to 0$ splits,

$\Rightarrow M$ is projective (P100: prop 3.3)

(v) $\Rightarrow$ (i): If I is a ideal of R,

$0 \to I \to R \to R/I \to 0$ exact,

$I, R, R/I \in {}_R\mathrm{Mod}$, $I, R, \underline{R/I}$ projective,

$\Rightarrow$ that sequence splits $\Rightarrow I$ is a direct summand of R

Every submodule of R is some ideal, I is arbitrary, R is semisimple. $\square$.

Opposite ring

Definition

$(R, +, \cdot)$ is a ring
$(R^{\text{op}}, +, \cdot^{\text{op}})$: the opposite ring
$\cdot^{\text{op}}$ defined as: $r_1 \cdot^{\text{op}} r_2 = r_2 \cdot r_1$

Enveloping algebra

Definition

k: commutative ring
L: k-algebra, commutative
$L^{\text{op}} = L$: because L is commutative
$L^e := L \otimes L^{\text{op}} = L \otimes L$: the enveloping "group" of L. The operation is $+$.
Define $\times$: $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.
Define scalar: $r(a \otimes b) = (ra) \otimes b = a \otimes (br)$.
L^e is an algebra, enveloping algebra.

We use L^e as ring.

finite separable extension and projective

Theorem

If L and k are fields and L is a finite separable extension of k, then L is a projective L^e-module, where L^e is the enveloping algebra.

Theorem: If L/k is a finite separable extension, then $L \in L^e\text{-Mod}$ is projective.

Proof:

- $L \in {}_L\text{Mod}_L \Rightarrow L \in L^e\text{-Mod}$ ✓
- It is enough to show $L^e = L \otimes_k L$ is a direct product of fields, then L^e is semisimple, $L \in L^e\text{-Mod}$ is projective.
- Since L/k is a finite separable extension, by Primitive Element theorem, $\exists\, \alpha \in L$, s.t. $L = k(\alpha)$.
- If $f(x) \in k[x]$ is a irreducible polynomial of α, we have

$$0 \longrightarrow (f) \xrightarrow{i} k[x] \xrightarrow{v} \underset{k(\alpha)}{\overset{\|}{L}} \longrightarrow 0 \quad \text{exact.}$$

- k is a field, $L \in k\text{-Mod}$ is free $\Rightarrow$ L is projective $\Rightarrow$ L is flat.
- then $0 \to L\otimes(f) \xrightarrow{1\otimes i} L \otimes k[x] \xrightarrow{1\otimes v} \underset{L^e}{\overset{\|}{L\otimes L}} \to 0$ exact.
- Let $L[y]$ be a polynomial ring, define

$$\theta: L \otimes k[x] \longrightarrow L[y] \qquad \text{isomorphism.}$$
$$a \otimes g(x) \longmapsto a g(y)$$

then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\otimes(f) & \xrightarrow{1\otimes i} & L\otimes k[x] & \longrightarrow & L^e & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & (f) & \longrightarrow & L[y] & \longrightarrow & L[y]/(f) & \longrightarrow & 0 \end{array}$$

Hence, $L^e \cong L[y]/(f)$. (P89/Prop2.70)

- L/k is separable, so $f(y) = \prod_i p_i(y)$, and $(f) = (p_1) \cap (p_2) \cap \cdots \cap \cdots$
- By Chinese Remainder theorem, $R/I_1\cap\cdots\cap I_k \cong R/I_1 \times \cdots \times R/I_k$

$$L^e \cong L[y]/(p_1)\cap(p_2)\cap\cdots \cong L[y]/(p_1) \times L[y]/(p_2) \times \cdots$$

$L[y]/(p_1)$ ↑ field, $L[y]/(p_2)$ ↑ field : (p_i): maximal ideal

□.

Definition of von Neumann regular rings

Definition

A ring R is von Neumann regular, if: $\forall r \in R$, there $\exists r' \in R$, s.t., $rr'r = r$.

Examples

Example

Finitely generated ideal is principal

Lemma

If R is a von Neumann regular ring, then every finitely generated left / right ideal is principal, and it is generated by an idempotent.

Lemma: If R is a von Neumann regular ring, then every finitely generated left / right ideal is principal, and it is generated by an idempotent.

Proof: Denote principal left ideal as $Ra = \{ra \mid r \in R\}$.

$\exists\, a'$, s.t. $a = aa'a$, we have $a'a$ is the idempotent,

$(a'a)^2 = a'\underline{aa'a} = a'a$.

And, $a = aa'a = a(a'a) \in Ra'a \Rightarrow Ra \subseteq Ra'a$

$a'a \in Ra \Rightarrow Ra'a \subseteq Ra$.

$\therefore Ra = Ra'a$.

- To prove every finitely generated left ideal is principl, it suffices to prove $Ra + Rb$ is principal.
- $Ra = Ra'a =: Re$, We claim that $Ra + Rb = Re + Rb = Re + Rb(1-e)$

  $e \in Re + Rb(1-e)$ ✓ $\quad b = b\cdot e + 1\cdot b(1-e) \in Re + Rb(1-e)$ ✓

  $e \in Re + Rb$ ✓ $\quad b(1-e) = (-b)\cdot e + 1\cdot b \in Re + Rb$ ✓

~~• But, Ra+Rb = Re+Rb~~

- $\exists\, f$, satisfies $f^2 = f$, s.t. $Rb(1-e) = Rf$.
- But, $Ra + Rb = Re + Rb = Re + Rb(1-e) = Re + Rf$ $= Re + Rg$

  $\overset{?}{=} R(e+f)$

  $e^2 = e$
  $g^2 = g$
  $eg = 0$, $ge = 0$.

$R(e+f) \subseteq Re + Rf$ ✓

$Re + Rf \subseteq R(e+f)$? $\quad (r_1 e + r_2 f)(e+f)$

$= r_1 e^2 + r_1 ef + r_2 fe + r_2 f^2$

$= r_1 e + \underbrace{r_1 ef + r_2 fe}_{?} + r_2 f$

- Define $g = (1-e)f$
- Check: $g^2 = g$ ✓

  $g^2 = (1-e)f(1-e)f = (1-e)(f - fe)f = (1-e)f^2$
  $= (1-e)f = g$.

  check $ge = 0$ ✓

  check $eg = 0$ ✓

$Ra + Rb = Re + Rg = R(e+g)$ ✓

□.

The main theorem (Harada)

Theorem

A ring R is von Neumann regular if and only if every right R-module is flat.

Theorem (Harada): A ring R is von Neumann regular iff every $M \in \mathrm{Mod}_R$ is flat.

Proof: "$\Rightarrow$": R is von Neumann, $B \in \mathrm{Mod}_R$,

P139: prop 3.60

we have $0 \to K \to F \to B \to 0$ exact,

it is enough to show, for any finitely generated ideal I, $K \cap FI = KI$.

- $KI \subseteq K \cap FI$ ✓
- $K \cap FI \subseteq KI$: By lemma, I is principal,

denote as $I = Ra$,

take $k \in K \cap FI$, $k \in K$, and $\exists f$, s.t.

$k = fa \in Fa = FRa = FI$.

then we have $k = fa = faa'a = ka'a \in Ka = KRa = KI$.

"$\Leftarrow$": For any $a \in R$, try to find the a':

Every module is flat, R flat, R/aR flat,

then $0 \to aR \to R \to R/aR \to 0$ exact.

R is free,

hence $aR \cap RI = aRI$ for any f.g. ideal I

Take $I = Ra$:

$aR \cap Ra = aRa$.

$a \in aR \cap Ra = aRa$, means,

there is a $a' \in R$, s.t. $a = aa'a$.

$\square$.

Relations between these two rings

Corollary

Every semisimple ring is von Neumann regular.

Summary